数理解析研究所講究録
963 巻 1996 年 90-96

ON MAXIMUM MODULUS OF ANALYTIC FUNCTIONS

SHIGEYOSHI OWA (近畿大学・理工)
WEI-QI YANG (北京理工大学)

ABSTRACT. Let A be the class of functions $f(z)$ analytic in $|z| < R$ with $f(0) = 1$. The object of the present paper is to investigate analyticity conditions of $M(r)$ which is the maximum modulus of $f(z)$ in A. Our results proved here provide a generalization of some results due to W. K. Hayman (J. Analyse Math. 1(1951), 135 - 154).

1. INTRODUCTION. Let A be the class of functions $f(z)$ analytic in the disk $|z| < R$ with $f(0) = 1$. The maximum modulus $M(r)$ given by

$$M(r) = \max\{f(z) \in A: z = re^{i\theta},\ 0 < r < R,\ 0 \leqq \theta \leqq 2\pi\} \tag{1.1}$$

was investigated by Hadamard [1], Hayman [3], et al. (see [2], [4]).

The curves from the origin to $|z| = R$ consisting of all points $z = re^{i\theta}$ which satisfy

$$\frac{\partial |f(re^{i\theta})|}{\partial \theta} = 0 \tag{1.2}$$

are called quasi-extremal curves of $f(z) \in A$. Some of quasi-extremal curves are called extremal curves of $f(z) \in A$ if

$$|f(re^{i\theta})| = M(r). \tag{1.3}$$

In the present paper, we show analyticity conditions of the maximum modulus $M(r)$ of $f(z) \in A$.

1991 AMS Subject Classification: Primary 30C45.

2. MAIN RESULTS. In order to discuss our problems, we have to use the following lemmas.

LEMMA 1 (Hayman [3]). Let $f(z)$ be analytic on the circle $|z| = r$. If it satisfies

$$(2.1) \qquad \frac{\partial |f(re^{i\theta_0})|}{\partial \theta} = 0, \qquad f(re^{i\theta_0}) \neq 0,$$

then $g(z) = zf'(z)/f(z)$ is real at the point $z_0 = re^{i\theta_0}$ and

$$(2.2) \qquad g(z_0) = r \frac{d}{dr} \log|f(\alpha(r))|,$$

where $\alpha(t)$ is the parametric representation with $t = |\alpha(t)|$ of any given curve which is analytic at a point z_0 and non-tangential with the circle $|z| = r$.

LEMMA 2. Let Γ be an analytic curve and intersect every circle $|z| = r$ $(0 < r \leqq R)$ at only one point. Then Γ is of the form

$$(2.3) \qquad \Gamma = \{z\colon\ z = re^{i\theta(r)},\ 0 \leqq r \leqq R\},$$

where $\theta(r)$ is a real analytic function.

PROOF. Since Γ is a simple analytic curve, there exists a parametric representation of Γ

$$(2.4) \qquad \Gamma = \{z\colon\ z = \alpha(t),\ 0 \leqq t \leqq 1\}$$

such that $\alpha(t)$ is an analytic injection with $\alpha(0) = 0$. Letting

$$(2.5) \qquad \log\frac{\alpha(t)}{t} = \phi(t) + i\psi(t),$$

we see that $\phi(t)$ and $\psi(t)$ are real analytic. Further, we know that

$$(2.6) \qquad r = te^{\phi(t)}$$

is exactly increasing in $t \in [0,1]$, and its inverse function is $t = t(r)$. Defining $\theta(r)$ by

$$(2.7) \qquad \theta(r) = \psi(t(r)),$$

we obtain the representation (2.4) of Γ.

Now we prove

THEOREM 1. If there exists a quasi-extremal curve

$$\Gamma = \{z: z = \alpha(r),\ 0 \leqq r < R\}$$

of $f(z)$ which is in the class A with modulus r such that $^\forall z_0 \in \Gamma$, $g(z_0)$ is the maximum real value of $g(z) = zf'(z)/f(z)$ on the circle $|z| = |z_0|$, then Γ is an extremal curve of $f(z)$ and $M(r)$ is real analytic on $[0,R)$.

PROOF. We may assume that $f(z)$ is analytic on the closed disk $|z| \leqq R$, other wise we consider a closed subdisk of $|z| < R$. Divide the disk $|z| < R$ into several annular domain D_k given by

$$D_k = \{z: r_{k-1} < |z| < r_k;\ r_0 = 0;\ r_n = R;\ k = 1,2,3,\ldots,n\}$$

such that $f(z) \neq 0$ in every D_k. It is known that there are 2^{n_k} quasi-extremal arcs

$$C_{kj} = \{z: z = \alpha_{kj}(r),\ j = 1,2,3,\ldots,2^{n_k}\}$$

of $f(z)$ in D_k, which are analytic in (r_{k-1}, r_k). Applying Lemma 1, we know that

$$(2.8) \qquad g(\alpha(r)) = r\frac{d}{dr}\log|f(\alpha(r))|$$

and

$$(2.9) \qquad g(\alpha_{k,j}(r)) = r\frac{d}{dr}\log|f(\alpha_{k,j}(r))|.$$

Then we have

$$(2.10) \qquad \log|f(\alpha(r))| = \int_0^r g(\alpha(t))\frac{dt}{t} \qquad (0 \leqq r < R)$$

and

$$(2.11) \qquad \log|f(\alpha_{1j}(r))| = \int_0^r g(\alpha_{1j}(r))\frac{dt}{t} \qquad (0 \leqq r < r_1),$$

which give

$$|f(\alpha_{1j}(r))| \leqq |f(\alpha(r))| \qquad (0 \leqq r < r_1). \tag{2.12}$$

Also the inequality (2.12) is still true for $r = r_1$. Therefore we obtain

$$M(r) = |f(\alpha(r))| \qquad (0 \leqq r \leqq r_1). \tag{2.13}$$

For $r_1 < r' < r'' < r_2$, it follows from (2.9) and (2.9) that

$$\frac{|f(\alpha_{2j}(r''))|}{|f(\alpha_{2j}(r'))|} = \exp\left\{ \int_{r'}^{r''} g(\alpha_{2j}(r)) \frac{dt}{t} \right\} \tag{2.14}$$

$$\leqq \exp\left\{ \int_{r'}^{r''} g(\alpha(t)) \frac{dt}{t} \right\}$$

$$= \frac{|f(\alpha(r''))|}{|f(\alpha(r'))|},$$

so

$$\frac{|f(\alpha_{2j}(r''))|}{|f(\alpha(r''))|} \leqq \frac{|f(\alpha_{2j}(r'))|}{|f(\alpha(r'))|}. \tag{2.15}$$

Letting $r' \to r_1$ in (2.15), we see that

$$|f(\alpha_{2j}(r''))| \leqq |f(\alpha(r''))|, \tag{2.16}$$

which proves that

$$M(r) = |f(\alpha(r))| \qquad (0 \leqq r \leqq r_2). \tag{2.17}$$

To do the above proces again and again, it follows that Γ is an extremal curve of $f(z)$ and $M(r)$ is real analytic on $[0,R)$.

Next, we derive

THEOREM 2. If $g(z)$ is analytic and univalent in $|z| < R$, and if there exists a quasi-extremal curve of $f(z) \in \mathcal{A}$ which intersects every circle $|z| = r$ $(0 < r < R)$ at only one point, then $M(r)$ is real analytic.

PROOF. Because $g(z)$ maps every quasi-extremal curve of $f(z)$ onto a half open interval on the real axis with the origin as an end point, the intersection of the real axis and the range of $g(z)$ is an open interval L which is divided by the origin into two intervals L_1(left hand interval) and L_2(right hand interval). These inverse images Γ_1 and Γ_2 are two quasi-extremal curves of $f(z)$ which are analytic curves.

Using Lemma 1 and Lemma 2, we have

$$\Gamma_j = \{z: z = re^{i\theta_j(r)}, \ j = 1,2\}, \tag{2.18}$$

where $\theta_j(r)$ are real analytic and

$$g(re^{i\theta_j(r)}) = r \frac{d}{dr} \log|f(re^{i\theta_j(r)})|. \tag{2.19}$$

Then we have

$$|f(re^{i\theta_j(r)})| = \exp\left\{ \int_0^r g(te^{i\theta_j(t)}) \frac{dt}{t} \right\}. \tag{2.20}$$

Since

$$g(re^{i\theta_1(r)}) < 0 < g(re^{i\theta_2(r)}), \tag{2.21}$$

it follows that

$$M(r) = |f(re^{i\theta_2(r)})| \tag{2.22}$$

which is real analytic.

Further, we prove

THEOREM 3. Under the assumption in Theorem 2, there exists a certain open disk $|z| < R_1$ such that $M(r)$ is the maximum modulus of its complex analytic extension $M(z)$ in $|z| < R_1$.

PROOF. Let D be the somain in which $M(z)$ is analytic. Then the origin is contained in D, and there exists a open disk $|z| < R_1$ in D such that, for any $|z| < R_1$,

$$(2.23) \qquad |ze^{i\theta_2(z)}| < R ,$$

where $\theta_2(z)$ is the complex analytic extension of $\theta_2(r)$. From (2.19), (2.21), and (2.22), we know that

$$(2.24) \qquad \mu(r) = r\frac{M'(r)}{M(r)} = g(re^{i\theta_2(r)})$$

is the maximum real value of $g(z)$ on the circle $|z| = r$, and $\mu(r)$ is exactly increasing and real analytic. It follows from (2.24) that

$$(2.25) \qquad \mu(z) = z\,\frac{M'(z)}{M(z)} = g(ze^{i\theta_2(z)})$$

and

$$(2.26) \qquad \mu(\bar{z}) = \overline{\mu(z)} \ , \quad M(\bar{z}) = \overline{M(z)} \ , \ \theta_2(\bar{z}) = \overline{\theta_2(z)} \ .$$

Suppose that $M(r)$ is not the maximum modulus of $M(z)$ in $|z| < R_1$, that is, there exists r_0 $(0 < r_0 < R_1)$ such that $M(r_0)$ is not the maximum modulus of $M(z)$ on the circle $|z| = r_0$. Then there is a point $z_0 = r_0e^{i\theta_0}$ such that

$$(2.27) \qquad \mu(\overline{z_0}) = \mu(z_0) > \mu(r_0)$$

by Theorem 1. This gives us that

$$(2.28) \qquad g(\overline{z_0}\ e^{i\theta_2(\overline{z_0})}) = g(z_0e^{i\theta_2(z_0)}) > \mu(r_0).$$

But, since

$$(2.29) \qquad |\overline{z_0}\ re^{i\theta_2(\overline{z_0})}||z_0e^{i\theta_2(z_0)}| = r_0^2\ |e^{i(\theta_2(\overline{z_0})+\theta_2(z_0))}| = r_0^2 ,$$

(2.28) contradicts the fact that $\mu(r)$ is the maximum real value on the circle $|z| = r$ and exactly increasing. This completes the proof of Theorem 3.

ACKNOWLEDGMENTS. This work of the first author was supported, in part, by the Japanese Ministry of Education, Science and Culture under Grant-in-Aid for General Scientific Research.

REFERENCES

[1] M. Hadamard, Sur les fonctions entiéres, Bull. Soc. Math. France 24(1896), 186 - 187.

[2] M. O. Bulmenthal, Sur le mode de croissance des fonctions entiéres, Bull. Soc. Math. France 35(1907), 213 - 232.

[3] W. K. Hayman, A characterization of the maximum modulus of functions regular at the origin, J. Analyse Math. 1(1951), 135 - 154.

[4] B. Anderson, On equivalent analytic functions, Arkiv Math. 1(1949), 77 - 92.

S. Owa: Department of Mathematics
Kinki University
Higashi-Osaka, Osaka 577
Japan

W. Yang: Department of Mathematics
Beijing Institute of Technology
Beijing 100081
People's Republic of China